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DETERMINATION OF THE AVERAGE CHARACTERISTICS OF ELASTIC FRAMEWORKS*

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A method is proposed for the approximate calculation of the average elastic characteristics of fine-celled framework structures of periodic configuration. The method is based on approximation of the "cell problem" of the theory of averaging /1—4/ by problems on the deformation of appropriate structures of beam, shell, etc., types. It is shown that the approximate values obtainable for the average characteristics and the solution of their appropriate problems are distinguished from the exact solutions by a quantity determined only by the error of the model being used. Examples are considered, namely, beam and box frameworks, and the construction of a framework with negative Poisson's ratios.

Methods for the average description of bodies containing a large number of fine vacancies /1, 2/ enable the structure of perodic configuration to be replaced by the consideration of continuous bodies similar in mechanical behaviour but with so-called average characteristics. The problem of finding the average characteristics is reduced in /2/ to the so-called cell problem of elasticity theory whose solution is quite difficult. At the same time, the solution of the cell problem in framework structures whose periodic element is a beam- or shell-type structure can be obtained by approximate methods to any accuracy, which is governed merely by the selection of the model.

An elastic structure of periodic configuration with perodicitiyy cell (PC) in the form of a parallelepiped $P_\varepsilon = \varepsilon P_1 = \{\varepsilon x : x \in P_1\}$ is considered, where $P_1 = \{x \in R^n: -\mu_i/2 \leqslant x_i \leqslant \mu_i/2, i = 1, \ldots, n\}$ $(n = 2, 3)$ is a rectangular parallelepiped with a characteristic length of the sides equal to one $(\mu_i \sim 1)$. The elastic material does not occupy the whole PC P_ε but only a part K_ε, which can be represented in the form $K_\varepsilon = \varepsilon K_1$. Under the condition that the characteristic (absolute or relative) PC dimension $\varepsilon \to 0$, production of the average is possible /2/. To determine the average elastic constants $\{\bar{a}_{ijkl}\}$ of a medium formed on the basis of the PC P_ε the part K_ε occupied by a material with the elastic constants $\{a_{ijkl}\}$ should minimize the functional /2/

$$F(\mathbf{u}) = \frac{1}{\operatorname{mes} P_1} \int_{K_1} a_{ijkl} (\operatorname{def} \mathbf{u})_{ij} (\operatorname{def} \mathbf{u})_{kl}\, dx \tag{1}$$

$$(\operatorname{def} \mathbf{u})_{ij} = {}^1/_2 (u_{i,j} + u_{j,i})$$

in the set of functions $\{H_2^1(P_1)\}^n$ under the additional conditions

$$\int_{P_1} \mathbf{u}(\mathbf{x})\, d\mathbf{x} = 0 \tag{2}$$

$$\mathbf{u} - {}^1/_2 (x_\alpha \mathbf{e}_\beta + x_\beta \mathbf{e}_\alpha) \in \Pi_1 \tag{3}$$

Here and henceforth, Π_1 is a class of functions periodic in P_1 ($\mathbf{e}_\alpha, \mathbf{e}_\beta$ are basis unit

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vectors of the Cartesian coordinate system). Afterwards, the average elastic constants are expressed in terms of the solution of problem (1)–(3) denoted by $\mathbf{u}^{\alpha\beta}$ (the indices α, β are from condition (3)) /2/

$$\bar{a}_{\alpha\beta\gamma\delta} = \frac{1}{\operatorname{mes} P_1} \int\limits_{K_1} a_{ijkl} (\operatorname{def} \mathbf{u}^{\alpha\beta})_{ij} (\operatorname{def} \mathbf{u}^{\gamma\delta})_{kl}\, d\mathbf{x} \tag{4}$$

It is convenient to transfer to the differential mode of writing the problem (1)–(3). The Euler equation for (1)–(3) has the form

$$\int\limits_{K_1} [a_{ijkl} (\operatorname{def} \mathbf{u})_{kl}],_j \varphi_i\, d\mathbf{x} + \int\limits_{\partial P_1 \cap \partial K_1} a_{ijkl} (\operatorname{def} \mathbf{u})_{kl} n_j \varphi_i\, d\mathbf{x} + \int\limits_{\partial P_1 \setminus (\partial P_1 \cap \partial K_1)} a_{ijkl} (\operatorname{def} \mathbf{u})_{kl} n_j \varphi_i\, d\mathbf{x} = 0 \tag{5}$$

for any function $\boldsymbol{\varphi} \in \{H_2^1 (P_1)\}^n$ satisfying condition (2) and the condition that follows from (3)

$$\boldsymbol{\varphi}(x) \in \Pi_1 \tag{6}$$

By virtue of the arbitrariness of the values of the function $\boldsymbol{\varphi}(x)$ within the domain K_1 it follows from (5) that in this domain the function $\mathbf{u}$ satisfies the equilibrium equation with zero mass forces: $[a_{ijkl} (\operatorname{def} \mathbf{u})_{kl}],_j = 0$. By virtue of the arbitrariness of the values of $\boldsymbol{\varphi}(x)$ in the domain $P_1 \setminus K_1$ we obtain that

$$\int\limits_{K_1} \mathbf{u}(\mathbf{x})\, d\mathbf{x} = 0 \tag{7}$$

We consider the integral with respect to $\partial P_1 \cap \partial K_1$ in (5) (i.e., over the common part of the boundaries of the PC P_1 and the domain K_1 occupied by the elastic material). Since the function $\boldsymbol{\varphi}(\mathbf{x})$ is periodic in P_1, the integral mentioned can be rewritten in the form

$$\sum_{j=1}^{n} \int\limits_{\Gamma_j \cap \partial K_1} [a_{ijkl} (\operatorname{def} \mathbf{u})_{kl} n_j (\mathbf{x}) + a_{ijkl} (\operatorname{def} \mathbf{u})_{kl} n_j (\mathbf{x} - \mu_j \mathbf{e}_j)]\, \varphi_i (\mathbf{x})\, d\mathbf{x} \tag{8}$$

where Γ_j is the face of the PC P_1 perpendicular to the axis Ox_j and intersecting it at $x_j = \mu_j/2$. The coordinates of the vector normal to the face Γ_j are $\{n_i (x)\} = \{0, \ldots, \delta_{ij}, \ldots, 0\}$ and of the vector normal to the opposite face are $\{0, \ldots, -\delta_{ij}, \ldots, 0\}$. On each of the faces Γ_j the set of traces of the functions from the space $\{H_2^1 (P_1)\}^n$ that satisfy (2) and (3) compact in the space $\{L_2 (\Gamma_j)\}^n$ /5/; consequently it follows from (8) that

$$a_{ijkl} (\operatorname{def} \mathbf{u})_{kl} n_j (\mathbf{x}) + a_{ijkl} (\operatorname{def} \mathbf{u})_{kl} n_j (\mathbf{x} - \mu_j \mathbf{e}_j) = 0$$

for $\mathbf{x} \in \Gamma_j \cap \partial K_1$. The quantities $\boldsymbol{\sigma}_n = \{a_{ijkl} (\operatorname{def} \mathbf{u})_{kl} n_j\}$ determine the stress vector. Hence

$$\boldsymbol{\sigma}_n (\mathbf{x}) + \boldsymbol{\sigma}_n (\mathbf{x} - \mu_j \mathbf{e}_j) = 0, \quad \mathbf{x} \in \partial\Gamma_j \cap \partial K_1 \tag{9}$$

The third term in (5) yields the following condition: the normal stresses are $\boldsymbol{\sigma}_n = \{a_{ijkl} (\operatorname{def} \mathbf{u})_{kl}\, n_j\} = 0$ on $\partial P_1 \setminus (\partial P_1 \cap \partial K_1)$ (i.e., on the part of the boundary of the domain K_1 that does not intersect the faces of the PC P_1). The formulation of the problem is obtained.

Let a cell structure (i.e., a structure occupying the domain K_1 of the PC P_1) be formed by elements having characteristic thicknesses of the order of $h_1, \ldots, h_n$ $(n = 2, 3)$ in the direction of the coordinate axes). Depending on whether one or two of the quantities h_i are small: $0 < h_i \ll \mu_i \sim 1$, we have a beam or plate (shell). The requirement that the h_i must be fixed, although small, non-zero numbers is essential here. Without the imposition of this condition, condition (4) in /2/, the sufficient condition for averaging, would not be satisfied. In practice, we can confine ourselves to the case when $10^{-3} \leqslant h_i/\mu_i \leqslant 10^{-1}$ /6–8/. The beam, plate, etc. theory problems that occur later are understood to be approximate solutions of the initial problem of elasticity theory with a certain accuracy α (in the norms to be mentioned later).

Average characteristics of a plane beam framework. Let us consider a plane framework with a PC P_1 of the type shown in Figs.1–3. Let the width of the elements forming the domain K_1 satisfy the condition $0 < h_i \ll \mu_i \sim 1$ and let the domain mentioned be occupied by an elastic material with Young's modulus E and Poisson's ratio ν. The solution of the elasticity theory problem of the deformation of such a structure is approximated by the solution of the problem of the deformation of a system of rigidly clamped beams (at their points of intersection) /7/ that generally operates under tension and bending. We will describe the beam deformation within the framework of the hypothesis of undeformed normals /6–8/.

We will construct a problem corresponding to the initial problem. Because the mass forces and stresses on the beam faces that do not intersect the PC P_1 equal zero, the displacements of each of the beams satisfy the well-known equilibrium equations with zero mass forces /6–8/. The corollaries of conditions (2), (3), (9) require a more detailed examination. We let v_α, w_α denote the displacements of points on the middle axis of the beam numbered with the subscript

α in the direction $\mathbf{e}_{1\alpha}$ tangent to the undeformed axis of the beam and the direction $\mathbf{e}_{2\alpha}$ normal to this same axis. The displacements $\mathbf{u}$ of points of the beam considered as a solid are related to $\{v_\alpha, w_\alpha\}$ by the hypothesis of the undeformed normal /6/: $\mathbf{u} \approx \mathbf{U} \equiv v_\alpha \mathbf{e}_{1\alpha} + w_\alpha \mathbf{e}_{2\alpha} + \xi (\mathbf{N}_\alpha - \mathbf{e}_{2\alpha})$, where $\mathbf{N}_\alpha$ is the normal direction to the deformed axis of the beam ($\mathbf{N}_\alpha = \mathbf{e}_{2\alpha} + w_\alpha' \mathbf{e}_{1\alpha}$ /6, 7/), $\xi \in [-h_\alpha/2, h_\alpha/2]$ is the coordinate across the beam axis.

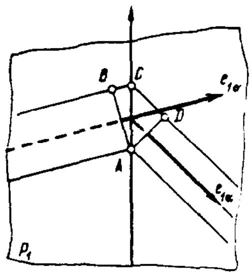

Fig.1

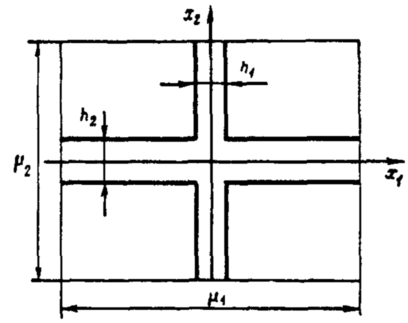

Fig.2

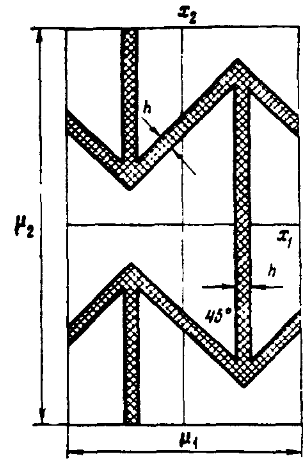

Fig.3

In the case under consideration condition (2) results in the condition

$$\sum_{\alpha=1}^{m} \int_{L_\alpha} (v_\alpha \mathbf{e}_{1\alpha} + w_\alpha \mathbf{e}_{2\alpha})\, ds = 0 \tag{10}$$

(m is the number of beams forming the cell structure). Integration is over the beam axes denoted by L_α, $\alpha = 1, \ldots, m$.

We examine the junction of beams belonging to adjacent PC or the ends of beams on opposite faces of the PC (Fig.1) by virtue of the periodicity of the problem under consideration. Condition (9) results in a deduction about the oppositeness of the forces on both sides of the line AC when the oppositeness of the normals to opposite faces of the PC P_1 is taken into account. Furthermore, the elements ABC and ACD are in equilibrium under the effect of zero external forces and strains in the sections AB, AC and AC, AD. We hence obtain that the strains in the sections AB and AD with the form $N_\alpha \mathbf{e}_{1\alpha} + Q_\alpha \mathbf{e}_{2\alpha}$, where N_α, Q_α are the tensile and transverse forces /7, 9/, are opposite. Since the moments on both sides of the line AC are opposite by virtue of (9), the moments in the sections AB and AD are opposite by virtue of the equilibrium condition for the elements ABC and ACD.

Furthermore, noting that the sections AB and AD are oriented oppositely (the direction $\mathbf{e}_{1\alpha}$ enters the element ABC and emerges from the element ACD, see Fig.1), we obtain the condition

$$N_\alpha \mathbf{e}_{1\alpha} + Q_\alpha \mathbf{e}_{2\alpha}, \quad M_\alpha \in \Pi_1 \tag{11}$$

The quantities $N_\alpha, Q_\alpha, M_\alpha$ mentioned, the tensile force, transverse force, and moment, are referred to identically oriented sections of the beam.

The kinematic conditions resulting from (3) have the form

$$v_\alpha \mathbf{e}_{1\alpha} + w_\alpha \mathbf{e}_{2\alpha} - {}^1/_2 h_\alpha w_\alpha' \mathbf{e}_{1\alpha} - {}^1/_2 (x_\gamma \mathbf{e}_\delta + x_\delta \mathbf{e}_\gamma), \quad w_\alpha' \mathbf{e}_{1\alpha} \in \Pi_1 \tag{12}$$

The peroidicity of the first function is obtained as a corollary to condition (2) in application to the point A (Fig.1). The periodicity of the second function is obtained by imposing a requirement about conservation of the magnitude of the angle between the beam axes during deformation (in the general case this condition cannot be derived from the hypothesis of undeformed normals since this hypothesis is not applicable in the domain $ABCD$).

Conditions (11) and (12) simplify considerably in the special case when the direction of the tangent to the undeformed beam axis is $\mathbf{e}_{1\alpha} \in \Pi_1$. In the case mentioned from (11) and (12) we have the condition

$$N_\alpha, \ Q_\alpha, \ M_\alpha, \ v_\alpha \mathbf{e}_{1\alpha} + w_\alpha \mathbf{e}_{2\alpha} - {}^1/_2 (x_\gamma \mathbf{e}_\delta + x_\delta \mathbf{e}_\gamma) \tag{13}$$
$$w_\alpha' \in \Pi_1, \quad \alpha = 1, \ldots, m$$

(since the sections AB, AD and AC coincide in this case). We note that if the tangent direction to the beam axis $\mathbf{e}_{1\alpha}$ on the faces of the PC P_1 is normal to these faces, the condition $w_\alpha' \in \Pi_1$ is already a direct consequence of (3) and the hypothesis of undeformed normals.

Approximate values of the elastic constants. The beam theory problem was considered above as a problem from whose solution a certain displacement field $\mathbf{U}$ approximating the true displacement field $\mathbf{u}$ determined from the solution of problem (1)–(3) can be constructed on the basis of the kinematic hypothesis taken. We denote the accuracy of this approximation (the error of the model) by α, i.e.,

$$\| \mathbf{u} - \mathbf{U} \| \equiv \| \operatorname{def} \mathbf{u} - \operatorname{def} \mathbf{U} \|_{L_2(P_1)} \leqslant \alpha \tag{14}$$

We introduce the quantity

$$A_{\alpha\beta\gamma\delta} = \frac{1}{\operatorname{mes} P_1} \int\limits_{K_1} a_{ijkl} (\operatorname{def} \mathbf{U}^{\alpha\beta})_{ij} (\operatorname{def} \mathbf{U}^{\gamma\delta})_{kl}\, dx \tag{15}$$

which it is natural to call approximate values (in the sense of their being used to calculate the functions $\mathbf{U}^{\alpha\beta}$) of the average elastic constants. The quantities $\{A_{\alpha\beta\gamma\delta}\}$ approximate the exact values of the average elastic constants $\{\bar{a}_{\alpha\beta\gamma\delta}\}$ given by (4) since by virtue of (4) and (15)

$$\begin{aligned} |\bar{a}_{\alpha\beta\gamma\delta} - A_{\alpha\beta\gamma\delta}| = \frac{1}{\operatorname{mes} P_1} \Big| \int\limits_{K_1} a_{ijkl} [(\operatorname{def} \mathbf{u}^{\alpha\beta})_{ij} (\operatorname{def} \mathbf{u}^{\gamma\delta})_{kl} - \\ (\operatorname{def} \mathbf{u}^{\alpha\beta})_{ij} (\operatorname{def} \mathbf{U}^{\gamma\delta})_{kl} + (\operatorname{def} \mathbf{u}^{\alpha\beta})_{ij} (\operatorname{def} \mathbf{U}^{\gamma\delta})_{kl} - \\ (\operatorname{def} \mathbf{U}^{\alpha\beta})_{ij} (\operatorname{def} \mathbf{U}^{\gamma\delta})_{kl}]\, dx \Big| \leqslant \\ \frac{1}{\operatorname{mes} P_1} |a_{ijkl}| (|\, \| \mathbf{U}^{\alpha\beta} \| \,| + \alpha)\alpha \equiv C (|\, \| \mathbf{U}^{\alpha\beta} \| \,| + \alpha)\alpha \\ (|\cdot| = \max_{ijkl} |\cdot|, \quad \max_{\alpha,\beta} |\cdot|) \end{aligned} \tag{16}$$

Remarks. 1°. As is seen from the formula obtained, to estimate the closeness of the exact and approximate values of the average elastic constants, it is necessary to have an estimate of the quantity $\| \mathbf{U}^{\alpha\beta} \|$. Such estimates can be obtained in specific cases, as is done below.

2°. For $\alpha = \gamma, \beta = \delta$, expressions (4) and (14) agree with twice the elastic strain energies corresponding to the displacement fields $\mathbf{u}^{\alpha\beta}$ and $\mathbf{U}^{\alpha\beta}$.

Estimate of the closeness between the solutions of the average problem and the problem with coefficients $\{A_{\alpha\beta\gamma\delta}\}$. According to (2), the displacements $\mathbf{v}_\varepsilon$ in the initial framework structure converge as $\varepsilon \to 0$ to the solution $\mathbf{v}$ of the elasticity theory problem of the deformation of a continuous medium with the elastic constants $\{\bar{a}_{\alpha\beta\gamma\delta}\}$ given by (4). This last problem has the well-known form

$$[\bar{a}_{ijkl} v_{k,l}]_{,j} = f_i, \quad \mathbf{v}|_{\partial Q} = \mathbf{v}^\circ \tag{17}$$

Convergence holds in the following sense /2/: let Q be the domain occupied by the framework structure (union of cells of the form P_ε), and Q_ε the domain occupied by the intrinsically elastic material (union of the domain K_ε). Then /2/

$$\| \mathbf{v}_\varepsilon - \mathbf{v} \|_{L_2(Q_\varepsilon)} = \delta(h_1, \ldots, h_n, \varepsilon) \tag{18}$$

where for any fixed $h_i > 0$ the quantity $\delta(h_1, \ldots, h_n, \varepsilon) \to 0$ as $\varepsilon \to 0$.

We consider the problem of the deformation of a medium with the elastic constants $\{A_{\alpha\beta\gamma\delta}\}$ defined above (that approximate $\{\bar{a}_{\alpha\beta\gamma\delta}\}$ to the accuracy of (16))

$$[A_{ijkl} V_{k,l}]_{,j} = f_i, \quad \mathbf{V}|_{\partial Q} = \mathbf{v}^\circ \tag{19}$$

As follows from /2/, the operator in (19) is positive-definite. We denote its ellipticity factor by $\beta(h_1, \ldots, h_n)$ (since β can depend and, as will be seen later, actually depends on $h_1, \ldots, h_n$). We esimate the difference between the solutions of problems (17) and (19). Compiling the problem for the difference between the solutions of (17) and (19) and multiplying the equation therein by $\mathbf{v} - \mathbf{V}$, we take account of (16) and obtain the estimate

$$\| \mathbf{v} - \mathbf{V} \|_{L_2(Q)} \leqslant \Lambda, \quad \Lambda = \frac{C(|\, \| \mathbf{U}^{\alpha\beta} \| \,| + \alpha) \| \mathbf{V} \| \alpha}{\beta(h_1, \ldots, h_n)} \tag{20}$$

It follows from (8) and (20) that

$$\| \mathbf{v}_\varepsilon - \mathbf{V} \|_{L_2(Q_\varepsilon)} \leqslant \| \mathbf{v}_\varepsilon - \mathbf{v} \|_{L_2(Q_\varepsilon)} + \| \mathbf{v} - \mathbf{V} \|_{L_2(Q)} \leqslant \delta(h_1, \ldots, h_n, \varepsilon) + \Lambda \tag{21}$$

Since $h_i > 0$ is a fixed number, by virtue of the recalled results /2/, we obtain from (21)

$$\lim_{\varepsilon \to 0} \| \mathbf{v}_\varepsilon - \mathbf{V} \|_{L_2(Q_\varepsilon)} \leqslant \Lambda \tag{22}$$

The estimates (16), (20)–(22) show that the approximate values of the elastic constants $\{\bar{a}_{\alpha\beta\gamma\delta}\}$ can generally be calculated by the proposed method to any accuracy determined only by the error in the model (the estimate (16)), where the solution of problem (19) will approximate the solution of the initial and average problems with an accuracy also determined just by the selection of the model (the estimates (20)–(22)).

Construction of the quantities $\{A_{\alpha\beta\gamma\delta}\}$ ensuring a given accuracy of the approximation of the elastic constants and the solutions is not trivial because of the dependence of the coefficients $\{A_{\alpha\beta\gamma\delta}\}$ and the quantities $\beta(h_1, \ldots, h_n)$, $\|U^{\alpha\beta}\|$, $\|V\|$ on $h_1, \ldots, h_n$. By establishing the dependence of the quantities listed on $h_1, \ldots, h_n$ in specific cases, conditions on the error of the model α can be obtained that are required for the evaluation of $\{\bar{a}_{\alpha\beta\gamma\delta}\}$ to a given accuracy.

Plane rectangular beam framework. We consider a framework whose PC P_1 is shown in Fig.2. We apply the method described above for the approximate calculation of the average framework characteristics. We note that the directions $e_{1\alpha}$ are periodic in P_1 in the case under consideration and conditions (11) and (12) can be taken in the form (13).

1°. $\alpha = \beta$. As can be seen conditions (13) correspond to the case of tension on the beam structure shown in Fig.2 in the direction of one of the coordinate axes. In the case $\alpha = \beta = 1$, the displacement field is $U^{11} = x_1 e_1$ in the horizontal beam while $U^{11} = 0$ in the vertical beam. According to (15), we obtain

$$A_{iiii} = \frac{E}{1-\nu^2}\frac{h_i}{\mu_i}, \quad i = 1, 2$$

The quantities h_i, μ_i are shown in Fig.2.

2°. $\alpha = 1, \beta = 2$ $(\alpha = 2, \beta = 1)$. Conditions (13) reduce to the following in the case under consideration:

$$[v_1 e_1 + w_1 e_2]_1 = \mu_1 e_2, \quad [v_2 e_2 + w_2 e_1]_2 = \mu_2 e_1 \tag{23}$$

w_i', w_i'', w_i''' are periodic in $[-\mu_i/2, \mu_i/2]$, $i = 1, 2$ $([f]_i = f(\mu_i/2) - f(-\mu_i/2))$.

The problem of bending (without tension) of the beam system shown in Fig.2 corresponds to conditions (23) for a symmetric structure (Fig.2). In addition to the equilibrium conditions

$$w_1^{IV} = 0, \quad w_2^{IV} = 0 \tag{24}$$

the equilibrium condition of an element lying at the intersection of the beams should be considered here. Since only moments (there is no beam tension) act on it $M_i = Eh_i^3[12(1-\nu^2)]^{-1} w_i''$, then

$$h_1^3(w_1''(+0) - w_1''(-0)) - h_2^3(w_2''(+0) - w_2''(-0)) = 0 \tag{25}$$

Because of symmetry the point of intersection of the beam axes does not experience displacement, i.e.,

$$w_i(+0) = w_i(-0) = 0, \quad i = 1, 2 \tag{26}$$

Moreover, the angles between the beam axes do not change during deformation

$$w_i'(+0) = w_i'(-0), \; i = 1, 2; \; w_1'(+0) = -w_2'(-0) \tag{27}$$

Solving problem (23)–(27), which is not difficult, and using Remark 2, we obtain

$$A_{1212} = \frac{4E}{1-\nu^2}\frac{h_1^3 h_2^3}{\mu_1\mu_2(\mu_1 h_2^3 + \mu_2 h_1^3)^2}\left[\frac{h_2^3}{\mu_1^2} + \frac{h_1^3}{\mu_2^2}\right] \tag{28}$$

The remaining elastic constants equal zero.

Summarizing, we obtain the governing relationships for the average medium (in which the constants $\{A_{\alpha\beta\gamma\delta}\}$ agree with the exact values of the average elastic constants $\{\bar{a}_{\alpha\beta\gamma\delta}\}$ to an accuracy determined by estimate (16))

$$\begin{aligned}\sigma_{ii} &= \frac{E}{1-\nu^2}\frac{h_i}{\mu_i}\varepsilon_{ii}, \quad i = 1, 2 \\ \sigma_{ij} &= \frac{4E}{1-\nu^2}\frac{h_1^3 h_2^3}{\mu_1\mu_2(\mu_1 h_2^3 + \mu_2 h_1^3)^2}\left[\frac{h_2^3}{\mu_1^2} + \frac{h_1^3}{\mu_2^2}\right]\varepsilon_{ij}, \quad i \neq j\end{aligned} \tag{29}$$

A more detailed estimate of the closeness between the approximate values of the elastic constants $\{A_{\alpha\beta\gamma\delta}\}$ and the solutions of the corresponding problem (19) to the exact values can be carried out in the case considered. As is seen from (29), the coefficients $\{A_{\alpha\beta\gamma\delta}\}$ and the ellipticity factor $\beta(h_1, h_2)$ are of the order of $1/Eh_i^3$. Consequently, the quantities $\|U^{\alpha\beta}\|$ and $\|V\|$ in (16), (20)–(22) are of the order of Eh_i^3. We hence obtain that the right side of the estimate (16) is of the order of $\alpha/(Eh_i^3)$ while the right side of the estimate (22) is of the order of $\alpha/(Eh_i^3)^2$. The possibility of obtaining the approximate values of the average elastic constants and the solutions of the appropriate problem by virtue of the estimates presented follow from the fact that the error α can be made less than any given quantity (in particular a quantity of the form h_i^m, $m \in N$) by selection of an appropriate model, while the order of the quantities $\beta(h_1, h_2)$, $\|U^{\alpha\beta}\|$, $\|V\|$ is conserved as the error α of the model diminishes.

As follows from the above, the hypothesis of undeformed normals that was used above can be utilized to compute the characteristics of a plane beam framework for $Eh_i^3 \gg 1$ (the PC K_1 of the structure is formed by fairly stiff elements) to the accuracy of order α. To obtain $\{A_{\alpha\beta\gamma\delta}\}$ with the required accuracy requires the use of more exact models in the general case (as is seen from before, the model possessing an accuracy $\alpha \sim h_i^2$ would solve the problem in the general case).

If α is understood to be the residual of the elastic strain energy in the modelling of the cellular structure, then the order of the right sides in (16), (22) is, respectively, α and $\alpha/(Eh_i^3)^2$.

The estimates (16), (20)–(22) enable us to deduce that as $\varepsilon \to 0$ the rectangular framework under consideration behaves as a solid elastic medium with governing Eqs.(29) to the accuracy determined above. By virtue of (29) orthotropy of the elastic constants is characteristic for the framework. Even for a PC with identical dimensions in the directions of the Ox_1 and Ox_2 axes, the orthotropy is conserved. The presence of an analogous effect is noted /4/ in the problem of the bending of a perforated plate. The coordinate axes are smooth for the tensor of the elastic constants $\{A_{\alpha\beta\gamma\delta}\}$ and the framework possesses a zero Poisson's ratio in these axes (to the accuracy determined by (16)). Moreover, since the quantities $h_i \ll 1$, then, as is seen from (29), the framework prossesses a shear modulus considerably less than its tensile modulus in these same axes.

Spatial rectangular beam framework. Let the PC P_1 be formed by rectangular bars lying on the coordinate axes and having a width h_i, height H_i, and end coordinates $\mu_i/2$, $-\mu_i/2$, $i = 1, 2, 3$. If $0 < h_i, H_i \ll \mu_i \sim 1$, but are here fixed, the cell problem is approximated by a problem on the deformation of a system of beams similar to that examined above. The approximate governing relationships have the form

$$\sigma_{ii} = \frac{E}{1-\nu^2}\frac{h_i H_i \mu_i}{\mu_1\mu_2\mu_3}\varepsilon_{ii}, \quad i = 1, 2, 3 \tag{30}$$
$$\sigma_{12} = \frac{4E}{1-\nu^2}\frac{h_1^3 h_2^3}{\mu_1\mu_2\mu_3(\mu_1 h_2^3 + \mu_2 h_1^3)^2}\left[\frac{H_2 h_2^3}{\mu_1^2} + \frac{H_1 h_1^3}{\mu_2^2}\right]\varepsilon_{12}$$

The equations connecting $\sigma_{13}, \varepsilon_{13}$ and $\sigma_{23}, \varepsilon_{23}$ are obtained from the equations for $\sigma_{12}, \varepsilon_{12}$ by permutation of the subscripts. As is seen from (30), the spatial framework retains the planar property noted above.

Frameworks of the types considered are utilized extensively as reinforcements in composite materials. In the case of fillers possessing low stiffness, the elastic characteristics of the composite are determined by the characteristics of the frame. Consequently, the low framework shear stiffness noted above results in smallness of the shear stiffness of the composites they bond, which is essential for the examination of composite plates and shells /10, 11/.

Boxlike framework. Let the PC P_1 of a three-dimensional framework have the form presented in Fig.2 in plane sections parallel to the coordinate plane Ox_1x_2, while the third coordinate is $x_3 \in [-\mu_3/2, \mu_3/2]$. If $0 < h_i \ll \mu_i \sim 1$. the cell problem is modelled by a problem on the deformation of two plates fixed rigidly along their line of intersection. The average governing equations (to the accuracy given by (17)) have the form

$$\sigma_{ii} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{h_i}{\mu_i}\left(\varepsilon_{ii} - \frac{\nu}{1-\nu}\varepsilon_{33}\right) \tag{31}$$
$$\sigma_{i3} = \frac{E}{1+\nu}\frac{h_i}{\mu_i}\varepsilon_{i3}, \quad i = 1, 2$$
$$\sigma_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left\{\left(\frac{h_1}{\mu_1} + \frac{h_2}{\mu_2}\right)\varepsilon_{33} - \frac{\nu}{1-\nu}\left(\frac{h_1}{\mu_1}\varepsilon_{11} + \frac{h_2}{\mu_2}\varepsilon_{22}\right)\right\}$$

The equations connecting $\sigma_{12}, \varepsilon_{12}$ agree with those presented in (29). As follows from (31), a boxlike framework in the plane Ox_1x_2 retains mainly the properties inherent in a plane framework. The framework behaves as an ordinary elastic material in the direction perpendicular to this plane: Poisson's ratio is positive, and the shear and tension moduli are of the same order.

Framework structure with negative Poisson's ratios. For a plane orthotropic continuous medium (when the principal axes coincide with the corodinate axes), the relation between the elastic constants $\{\bar{a}_{\alpha\beta\gamma\delta}\}$, Young's modulus E_i and Poisson's ratios ν_i has a well-known form /9/, in particular

$$E_1\nu_2/(1 - \nu_1\nu_2) = \bar{a}_{1122}/\bar{a}_{2222}$$

Since $\bar{a}_{2222}$, E_1, and $1 - \nu_1\nu_2$ are positive quantities /9/, the sign of Poisson's ratio ν_2 agrees with the sign of $\bar{a}_{1122}$. The same holds for Poisson's ratio ν_1 /9/.

We consider the problem of the deformation of a plane framework structure whose PC P_1 is displaced in Fig.3. The solution of this problem for $h_i \ll \mu_i \sim 1$ approximates the solution of the problem about the deformation of a system of rigidly clamped beams. Utilizing the hypothesis of the undeformed normal, we calculate the coefficient A_{1122}. The solution is simplified if the presence of definite symmetry in the cell structure is used. Namely, a quadrant of the PC P_1 can be considered (Fig.3). We consider the problem of the equilibrium of the element in question for zero mass forces with the conditions

$$\begin{aligned}
&v = w = \pm\frac{\mu_1}{4\sqrt{2}} \quad \text{at points} \quad \left(0, \frac{\mu_2}{4}\right), \left(\frac{\mu_1}{2}, \frac{\mu_2}{4}\right) \\
&v = w = 0 \quad \text{at the point} \quad \left(\frac{\mu_1}{4}, 0\right), \quad (\alpha = \beta = 1) \\
&v = w = 0 \quad \text{at the points} \quad \left(0, \frac{\mu_2}{4}\right), \left(\frac{\mu_1}{2}, \frac{\mu_2}{4}\right) \\
&v = -\frac{\mu_2}{4}, w = 0 \quad \text{at the point} \quad \left(\frac{\mu_1}{4}, 0\right) \quad (\alpha = \beta = 2) \\
&w'' = 0 \quad \text{at the points} \quad \left(0, \frac{\mu_2}{4}\right), \left(\frac{\mu_1}{2}, \frac{\mu_2}{4}\right) \quad \text{for all mentioned} \quad \alpha, \beta
\end{aligned} \tag{32}$$

and, moreover, the solution of the problem is symmetrical about the line $x_1 = \mu_1/4$.

It can be confirmed that merging the solutions of a problem of the kind mentioned for each cell structure quadrant yields the solution of the problem on the equilibrium of the structure displaced in Fig.3 with conditions (13) (the vector $\mathbf{e}_{1\alpha}$ is periodic). The solution of the problem about the equilibrium of a quadrant of a cell structure element with conditions (32) is constructed analytically. This easily reproducible solution is not presented because of its awkwardness. Calculations showed that for $\mu_1 = 2\sqrt{2}, \mu_2 = 4 - \sqrt{2}$ the coefficient

$$A_{1122} = \frac{4Eh}{(1-\nu^2)\mu_1\mu_2} \int v^{11'} v^{22'}\, ds + \frac{Eh^3}{3(1-\nu^2)\mu_1\mu_2} \int w^{11''} w^{22''}\, ds$$

takes a negative value (integration is along the axes of the cell structure quadrant). Therefore, the framework with PC P_1 shown in Fig.3 in the axes mentioned here, behaves as a solid elastic medium with negative Poisson's ratios ν_1, ν_2 to an accuracy given by the estimates (16), (20)–(22) as $\varepsilon \to 0$.

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